Calculus II - Day 4

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Divergence Test, p-test

Goals for today:

- Find a criterion satisfied by all convergent series and use this to conclude that certain series do not converge
- Determine for which p the sum
$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$
converges (and why!)

Reminders

- Gradescope HW #1: due Tuesday evening
- MyLab HW #3: due Wednesday at noon

Example: (A telescoping series)

$$\sum_{k=2}^{\infty} \left(\sin\left(\frac{\pi}{k}\right) - \sin\left(\frac{\pi}{k+1}\right) \right)$$

$$S_1 = a_2 = \sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{3}\right)$$

$$S_2 = a_2 + a_3 = \left(\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{3}\right) \right) + \left(\sin\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{4}\right) \right)$$

(Middle two $\sin\left(\frac{\pi}{3}\right)$ terms cancel out)

$$S_N = a_2 + a_3 + \cdots + a_{N+1}$$

$$\sum_{k=2}^{N+1} \left(\sin\left(\frac{\pi}{k}\right) - \sin\left(\frac{\pi}{k+1}\right) \right)$$

$$= \left(\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{3}\right)\right) + \left(\sin\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{4}\right)\right) + \dots$$

$$+ \left(\sin\left(\frac{\pi}{N}\right) - \sin\left(\frac{\pi}{N+1}\right)\right) + \left(\sin\left(\frac{\pi}{N+1}\right) - \sin\left(\frac{\pi}{N+2}\right)\right)$$

(All middle terms cancel out, leaving only the first and last)

$$\Rightarrow \sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{N+2}\right)$$

$$\sum_{k=2}^{\infty}\left(\sin\left(\frac{\pi}{k}\right) - \sin\left(\frac{\pi}{k+1}\right)\right) = \lim_{N\to\infty} S_N$$

$$= \lim_{N\to\infty}\left(\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{N+2}\right)\right)$$

$$= 1 - 0 = \boxed{1}$$

Bad news: Other than geometric and telescoping series, **it's hard to find a formula for** S_N, and therefore impossible to determine the sum's convergence (or even whether it converges).

Best we can do: Answer the question: Does $\sum_{k=1}^{\infty} a_k$ converge or not?

In order for there to be any hope of a series converging, the terms must approach 0.

The Divergence Test:

If $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$.
(If $\lim_{k\to\infty} a_k \neq 0$, then $\sum_{k=1}^{\infty} a_k$ diverges.)

Proof: Suppose $\sum a_k$ converges. Let

$$S_N = \sum_{k=1}^{N} a_k$$

be the Nth partial sum.

We know that:

$$\lim_{N\to\infty} S_N = S$$

and

$$\lim_{N\to\infty} S_{N-1} = S.$$

Thus,

$$\lim_{N\to\infty}(S_N - S_{N-1}) = \lim_{N\to\infty} S_N - \lim_{N\to\infty} S_{N-1} = S - S = 0.$$

So,

$$\lim_{N\to\infty} a_N = 0.$$

Example: The series:

$$\sum_{k=1}^{\infty}\left(1+\frac{1}{k}\right).$$

Here, $a_k = 1 + \frac{1}{k}$.

$$\lim_{k\to\infty}\left(1+\frac{1}{k}\right) = 1 \neq 0.$$

By the Divergence Test, this series diverges.

Example:

$$\sum_{k=1}^{\infty}\frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \quad \text{(Harmonic series)}$$

The terms do go to 0... but that doesn't mean the sum converges!

$$\Rightarrow \text{the Divergence Test is inconclusive.}$$

This series actually diverges:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots$$

Group terms:

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots$$

(Number of terms grouped together grows with powers of 2)

$$1 + \frac{1}{2} + \left(\frac{\mathbf{1}}{\mathbf{4}} + \frac{1}{4}\right) + \left(\frac{\mathbf{1}}{\mathbf{8}} + \frac{\mathbf{1}}{\mathbf{8}} + \frac{\mathbf{1}}{\mathbf{8}} + \frac{1}{8}\right) + \left(\frac{\mathbf{1}}{\mathbf{16}} + \dots + \frac{1}{16}\right) + \dots$$

(The bolded terms are lowered to a term less than the actual term)

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots = \infty$$

The series is "larger than" infinity, so the Harmonic series diverges:

$$\sum_{k=1}^{\infty}\frac{1}{k} = \infty$$

Example: What about

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots$$

$$= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{7^2} + \frac{1}{8^2} + \cdots + \frac{1}{15^2} + \ldots$$

(Group terms)

$$= \frac{1}{1^2} + \left(\frac{1}{2^2} + \frac{1}{3^2}\right) + \left(\frac{1}{4^2} + \cdots + \frac{1}{7^2}\right) + \left(\frac{1}{8^2} + \cdots + \frac{1}{15^2}\right) + \ldots$$

$$< 1 + \left(\frac{1}{2^2} + \frac{1}{2^2}\right) + \left(\frac{1}{4^2} + \cdots + \frac{1}{4^2}\right) + \left(\frac{1}{8^2} + \cdots + \frac{1}{8^2}\right) + \ldots$$

$$= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = 2$$

So, this series converges to a number less than 2.

Conclusion:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} < 2$$

Why does this mean the series converges?

Let $S_N = \sum_{k=1}^{N} \frac{1}{k^2}$.

The sequence $\{S_N\}$ is increasing (therefore monotonic) and bounded above by 2. By the Monotone Convergence Theorem, it converges:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \lim_{N \to \infty} S_N \text{ converges.}$$

Fact:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} = 1.644934\ldots$$

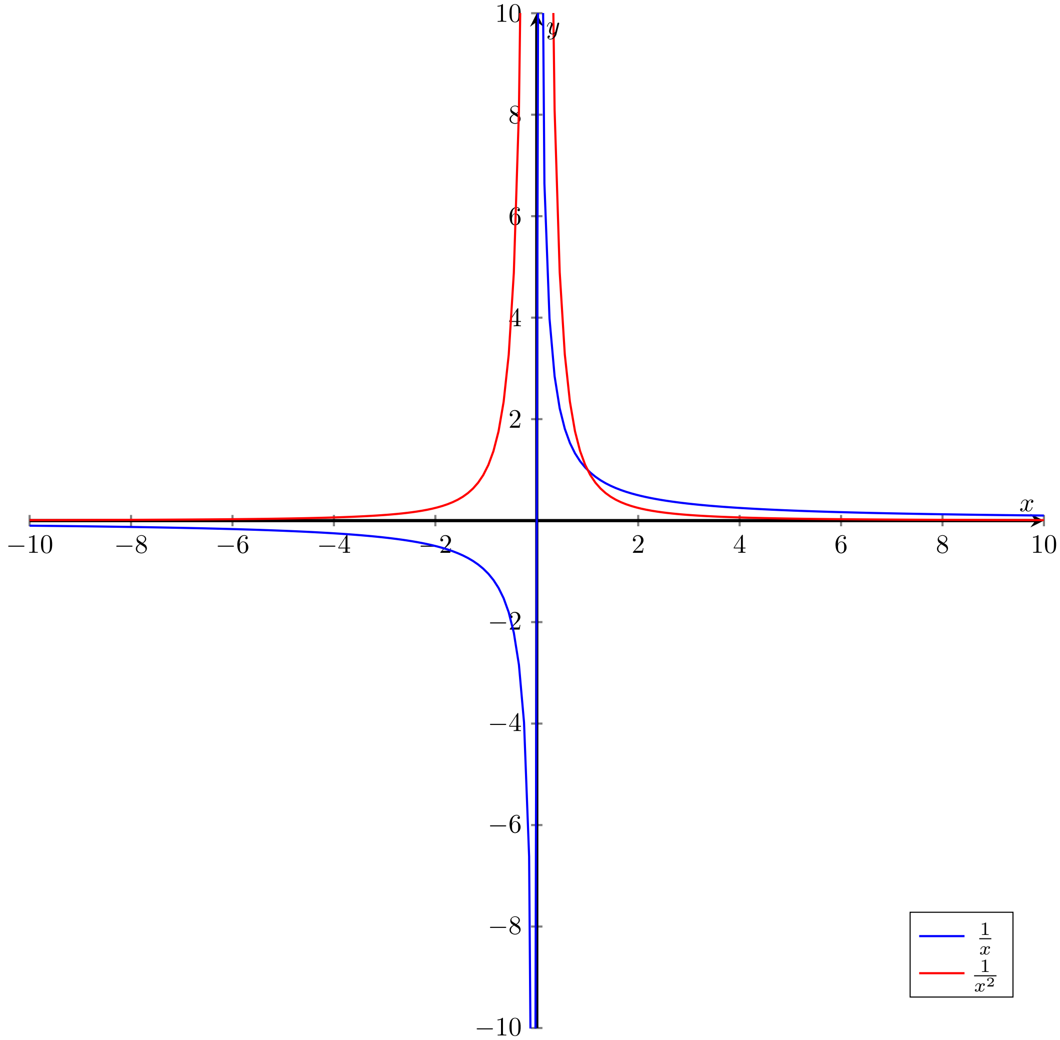
In summary:

$$\sum_{k=1}^{\infty} \frac{1}{k} = \infty \quad \text{but} \quad \sum_{k=1}^{\infty} \frac{1}{k^2} \text{ converges.}$$

What's the difference? The terms in both series go to 0, but the second series goes to 0 faster!

In order to converge, not only must $a_k \to 0$, but it must do so quickly!

Comparison of $\frac{1}{x}$ and $\frac{1}{x^2}$



Example:

$$\sum_{k=1}^{\infty} \frac{1}{k^{1.5}} \quad ? \quad \sum_{k=1}^{\infty} \frac{1}{k^{0.0001}} \quad ?$$

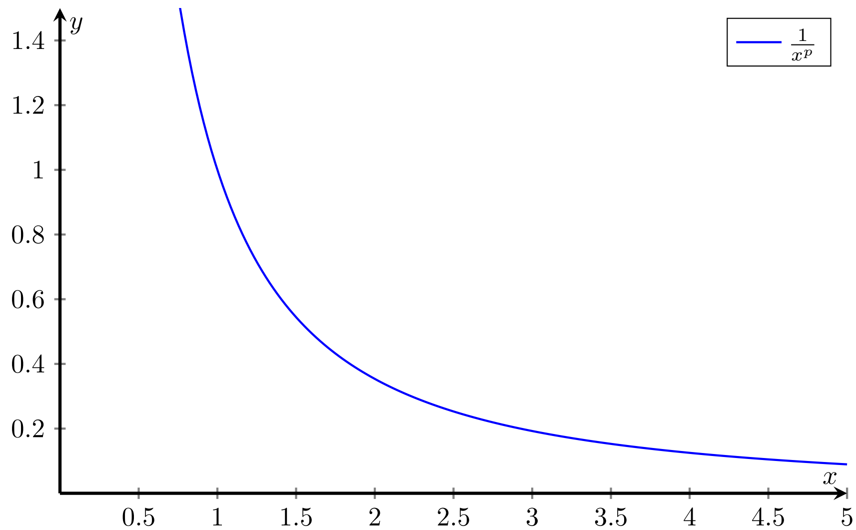
Fact: p-series test

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots$$

- Converges when $p > 1$
- Diverges when $p \leq 1$

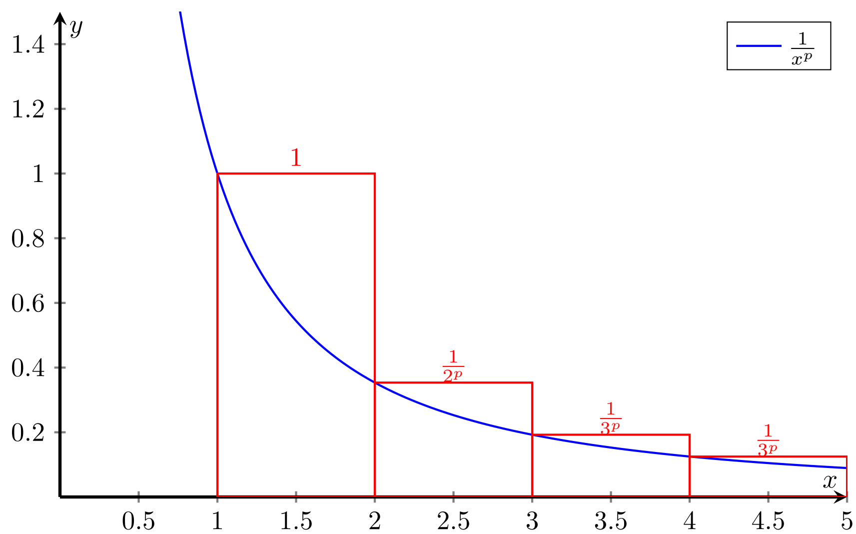
Proof: Suppose $p > 1$. Graph $f(x) = \frac{1}{x^p}$:

Graph of $f(x) = \frac{1}{x^p}$ for $p > 1$

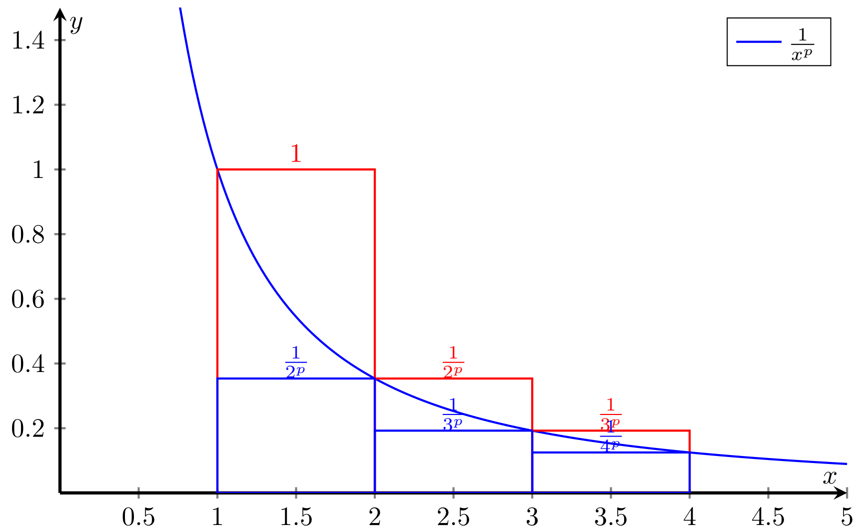


Now, we find the area under the curve using boxes.

$f(x) = \frac{1}{x^p}$ with boxes



$f(x) = \frac{1}{x^p}$ with red and blue boxes



In general:

$$\sum_{k=2}^{N} \frac{1}{k^p} < \int_1^N \frac{1}{x^p}\,dx < \sum_{k=1}^{N-1} \frac{1}{k^p}$$

$$\int_1^N \frac{1}{x^p}\,dx = \int_1^N x^{-p}\,dx = \frac{1}{-p+1} x^{-p+1}\Big|_1^N$$

$$= \frac{1}{-p+1} N^{-p+1} - \frac{1}{-p+1}$$

$$= \frac{1}{1-p} N^{-(p-1)} + \frac{1}{p-1}$$

$$\sum_{k=2}^{N} \frac{1}{k^p} < \frac{1}{1-p} N^{-(p-1)} + \frac{1}{p-1} < \sum_{k=1}^{N-1} \frac{1}{k^p}$$

$$S_{N-1} < \frac{1}{1-p} N^{-(p-1)} + \frac{1}{p-1} < S_{N-1}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$S - 1 \quad < \quad \frac{1}{p-1} \quad < \quad S$$

Rearrange:

$$\frac{1}{p-1} < S < \frac{1}{p-1} + 1$$

Therefore, the sum converges when $p > 1$:

$$\frac{1}{p-1} < \sum_{k=1}^{\infty} \frac{1}{k^p} < \frac{1}{p-1} + 1$$

Example (p=2):

$$\frac{1}{2-1} < \sum_{k=1}^{\infty} \frac{1}{k^2} < \frac{1}{2-1} + 1$$

$$1 < \sum_{k=1}^{\infty} \frac{1}{k^2} < 2$$

Example (p=3):

$$\sum_{k=1}^{\infty} \frac{1}{k^3} \text{ is between } \frac{1}{3-1} \text{ and } \frac{1}{3-1} + 1 \quad \left(\frac{1}{2} \text{ and } \frac{3}{2}\right)$$

Actual value: $1.2020569\ldots$ (Apery's constant)